Kuwait University
Department of Mathematics & Computer Science

Math 102 | July 12, 2003
Calculus II | First Midterm Examination | Time: 75 minutes

Use of calculators, mobile phones or pagers is not allowed during the exam.

1. Let $f(x) = \ln(\cos x^3) - 2x + 1$, where $0 \le x \le 1$. [2 pts each]

   a. Show that f^{-1} exists.

   b. Find the equation of the tangent line to the graph of f^{-1} at the point $P(1,0)$.

2. Find y' where [3 pts]

$$y = \frac{\sqrt[3]{x+1}\ \log_5(x^2+1)}{(1+\ln x)^2\ e^{1-\sqrt{x}}}$$

3. Prove the following identity: [3 pts]

$$2\sin^{-1} x = \cos^{-1}(1-2x^2), \qquad x \ge 0$$

4. Evaluate the following integrals: [4 pts each]

   a. $\displaystyle\int_0^{\ln 2} \sqrt{\cosh x - 1}\, dx$

   b. $\displaystyle\int \frac{x^{1/e} - e^{1/x}}{x^2}\, dx$

   c. $\displaystyle\int \frac{5^{\cot x}}{\sin^2 x}\, dx$

5. Evaluate the following limit if it exists: [3 pts]

$$\lim_{x\to\infty} (1+e^x)^{1/x}$$

1. $f'(x) = -\frac{\sin x^3}{\cos x^3} \cdot 3x^2 - 2 < 0, \quad 0 \le x \le 1$ (1)

$\Rightarrow f \downarrow$ so f^{-1} exists . (1)

$(0,1)$ lies on $y = f(x) \Rightarrow (1,0)$ lies on $y = g(x) = f^{-1}(x)$

$\therefore g(1) = 0$ and $g'(1) = \frac{1}{f'(0)} = \frac{1}{-2}$ (1) Hence tangent line at $(1,0)$ is $y - 0 = g'(1)(x-1)$ or $y = -\frac{1}{2}(x-1)$ (1)

2. $y = \frac{\sqrt[3]{x+1}\,\log_5(x^2+1)}{(1+\ln x)^2 \cdot e^{1-\sqrt{x}}}$

$\ln y = \frac{1}{3}\ln(x+1) + \ln\left(\frac{\ln(x^2+1)}{\ln 5}\right) - 2\ln(1+\ln x) - (1-\sqrt{x})$ (1)

$\frac{y'}{y} = \frac{1}{3(x+1)} + \frac{1}{\ln(x^2+1)} \cdot \frac{1}{(x^2+1)} \cdot 2x - \frac{2}{1+\ln x} \cdot \frac{1}{x} + \frac{1}{2}x^{-\frac{1}{2}}$ (2)

$\therefore y' = y[\cdots + \cdots - \cdots + \cdots]$.

3. Let $f(x) = 2\sin^{-1}x - \cos^{-1}(1-2x^2)$. Then $f'(x) = 0$ (2)

$f(x) = c$, $f(0) = 0 \quad \therefore c = 0 \quad \therefore f(x) = 0$. (1)

4 (a) $\int_0^{\ln 2} \sqrt{\cosh x - 1}\,dx = \int_0^{\ln 2} \frac{\sqrt{\cosh^2 x - 1}}{\sqrt{\cosh x + 1}}\,dx = \int_0^{\ln 2} \frac{\sinh x\,dx}{\sqrt{1+\cosh x}} = \left[2\sqrt{1+\cosh x}\right]_0^{\ln 2} = 3 - 2\sqrt{2}$ (1) (2) (1) (1) (2)

(b) $\int \frac{x^{1/e} - e^{1/x}}{x^2}\,dx = \int x^{1/e - 2}\,dx - \int \frac{e^{\frac{1}{x}}}{x^2}\,dx = \frac{x^{\frac{1}{e}-1}}{1/e - 1} + e^{1/x} + c$ (1) (2)

(c) $\int \frac{5^{\cot x}}{\sin^2 x}\,dx = -\int 5^u\,du$, where $u = \cot x$ (2)

$= \frac{-5^{\cot x}}{\ln 5} + c$

5. Let $y = (1+e^x)^{1/x}$, then $\ln y = \frac{1}{x}\ln(1+e^x)$ (1)

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(1+e^x)}{x} : \frac{\infty}{\infty}$ (1.5)

(L) (R) $= \lim_{x\to\infty} \frac{e^x}{1+e^x} = 1 \quad \therefore \lim_{x\to\infty} y = e$ (.5)

$2\sin^{-1}x = \cos^{-1}(1-2x^2)$.

$2\sin^{-1}x - \cos^{-1}(1-2x^2)$

$\cos 2\sin^{-1}x - \cos\cos^{-1}(1-2x^2)$

$\cos(y2)$

$\cos^2 x - \sin^2 x$